

DISTRIBUTIONAL CHAOS AND FACTORS

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ABSTRACT. We show the existence of a dynamical system without any distributionally scrambled pair which is semiconjugated to a distributionally chaotic factor.

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1. INTRODUCTION

Semiconjugacy is used as a common tool for proving topological chaos or positive topological entropy. The usual technique is to find a semiconjugacy π with a chaotic system and transfer the chaos to the extension. By continuity of π , the topological entropy of the extension is not smaller than the entropy of factor system. Unfortunately, semiconjugacy may not automatically guarantee the distributional chaos, which was introduced in [1]. Authors in [2],[3],[4] developed several techniques for proving distributional chaos via semiconjugacy, usually using a symbolic space as the factor space. Example in [2] shows the existence of distributionally chaotic factor which is semiconjugated to the system with no three-points distributionally scrambled sets. The aim of the paper is to improve this result and find a distributionally chaotic factor which has an extension without any distributionally scrambled pair.

2. TERMINOLOGY

Let (X, d) be a non-empty compact metric space. Let us denote by (X, f) the *topological dynamical system*, where f is a continuous self-map acting on X . We define the *forward orbit* of x , denoted by $Orb_f^+(x)$ as the set $\{f^n(x) : n \geq 0\}$. Let (X, f) and (Y, g) be dynamical systems on compact metric spaces. A continuous map $\pi : X \rightarrow Y$ is called a semiconjugacy between f and g if π is surjective and $\pi \circ f = g \circ \pi$. In this case we can say that (Y, g) is a factor of the system (X, f) or equivalently (X, f) is an extension of the system (Y, g) .

Definition 1. A pair of two different points $(x_1, x_2) \in X^2$ is called *scrambled* if

$$\liminf_{k \rightarrow \infty} d(f^k(x_i), f^k(x_j)) = 0 \quad (1)$$

and

$$\limsup_{k \rightarrow \infty} d(f^k(x_i), f^k(x_j)) > 0. \quad (2)$$

A subset S of X is called *scrambled* if every pair of distinct points in S is scrambled. The system (X, f) is called *chaotic* if there exists an uncountable scrambled set.

Definition 2. For a pair (x_1, x_2) of points in X , define the *lower distribution function* generated by f as

$$\Phi_{(x_1, x_2)}(\delta) = \liminf_{m \rightarrow \infty} \frac{1}{m} \# \{0 < k < m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

and the *upper distributional function* as

$$\Phi_{(x_1, x_2)}^*(\delta) = \limsup_{m \rightarrow \infty} \frac{1}{m} \# \{0 < k < m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A .

A pair $(x_1, x_2) \in X^2$ is called *distributionally scrambled of type 1* if

$$\Phi_{(x_1, x_2)}^* \equiv 1 \text{ and } \Phi_{(x_1, x_2)}(\delta) = 0, \text{ for some } 0 < \delta \leq \text{diam } X,$$

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distributionally scrambled of type 2 if

$$\Phi_{(x_1, x_2)}^* \equiv 1 \text{ and } \Phi_{(x_1, x_2)} < \Phi_{(x_1, x_2)}^*,$$

distributionally scrambled of type 3 if

$$\Phi_{(x_1, x_2)} < \Phi_{(x_1, x_2)}^*.$$

The dynamical system (X, f) is distributionally chaotic of type i (DC i for short), where $i = 1, 2, 3$, if there is an uncountable set $S \subset X$ such that any pair of distinct points from S is distributionally scrambled of type i .

3. DISTRIBUTIONAL CHAOS AND FACTORS

We will show the existence of a system without any distributionally scrambled pair which is semiconjugated to a distributionally chaotic factor. This system is three-dimensional union of countably many homocentric cylinders with unit height and converging radius. First we state the following technical lemma about rotation on circle. Let $u \in \mathbb{S}$ and $v \in \mathbb{S}$ be determined by normed angles $\phi_u \in I$ and $\phi_v \in I$. These points rotate along the circle by different angles $r_u \in I$, respectively $r_v \in I$, i.e.

$$\begin{aligned} \phi_u &\mapsto (\phi_u + r_u) \bmod 1 \\ \phi_v &\mapsto (\phi_v + r_v) \bmod 1. \end{aligned} \tag{3}$$

We denote the relative angle of rotation by $\Delta r = |r_u - r_v|$ and assume that the metric on \mathbb{S} is $\rho(\alpha, \beta) = \min\{|\alpha - \beta|, 1 - |\alpha - \beta|\}$.

Lemma 1. *For every number $\delta > 0$ and every integer $p > \frac{2}{\Delta r}$, the following estimation holds:*

$$\frac{1}{p} \# \{0 < i < p; \rho((\phi_u + ir_u) \bmod 1, (\phi_v + ir_v) \bmod 1) < \delta\} < 3\delta.$$

Proof. Because $\rho((\phi_u + ir_u) \bmod 1, (\phi_v + ir_v) \bmod 1) = \rho(\phi_u, (\phi_v + i\Delta r) \bmod 1)$, it is sufficient to show

$$\frac{1}{p} \# \{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \bmod 1) < \delta\} < 3\delta.$$

The expression $[p \cdot \Delta r]$ determines the number of turns that the point v makes along the circle by rotation through the angle Δr after p iterations, where $[x]$ denotes the integer part of x . The maximal number of iterations i during one turn, for which $\rho(\phi_u, (\phi_v + i\Delta r) \bmod 1) < \delta$, is $\frac{2\delta}{\Delta r}$. It follows

$$\frac{1}{p} \# \{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \bmod 1) < \delta\} < \frac{1}{p} ([p \cdot \Delta r] \frac{2\delta}{\Delta r} + \frac{2\delta}{\Delta r}) < 2\delta + \frac{2\delta}{p\Delta r}.$$

Because $p > \frac{2}{\Delta r}$, we can estimate the second term by δ , i.e. $\frac{2\delta}{p\Delta r} < \delta$. □

Theorem 1. *There exists a DC1 dynamical system (Y, f) which is semiconjugated to an extension (X, F) which possess no distributionally scrambled pair (of type 1 or 2).*

Proof. The space X is defined

$$X = \left(\left\{ \left(\left(2 - \frac{1}{k} \right) \cos 2\pi\phi, \left(2 - \frac{1}{k} \right) \sin 2\pi\phi \right) : k \in \mathbb{N}, \phi \in I \right\} \cup \{ [2 \cos 2\pi\phi, 2 \sin 2\pi\phi] : \phi \in I \} \right) \times I,$$

where I is the unit interval. Each point $u = [r_u \cos 2\pi\phi_u, r_u \sin 2\pi\phi_u, z_u]$ in X is determined by its angle $\phi_u \in I$, radius $r_u \in \{2 - \frac{1}{k} : k \in \mathbb{N}\} \cup \{2\}$ and height $z_u \in I$.

The space is endowed with max-metric

$$d(u, v) = \max\{|r_u - r_v|, |z_u - z_v|, \rho(\phi_u, \phi_v)\}, \tag{4}$$

where $\rho(\phi_u, \phi_v) = \min\{|\phi_u - \phi_v|, 1 - |\phi_u - \phi_v|\}$. We define the mapping $F : X \rightarrow X$ as identity on the limit cylinder,

$$[2 \cos 2\pi\phi, 2 \sin 2\pi\phi, z] \mapsto [2 \cos 2\pi\phi, 2 \sin 2\pi\phi, z],$$

and as a composition of rotation and continuous mapping g on inner cylinders,

$$\left(\left(2 - \frac{1}{k} \right) \cos 2\pi\phi, \left(2 - \frac{1}{k} \right) \sin 2\pi\phi, z \right) \mapsto \left(\left(2 - \frac{1}{k+1} \right) \cos 2\pi(\phi + \Psi(k, z)), \left(2 - \frac{1}{k+1} \right) \sin 2\pi(\phi + \Psi(k, z)), g_k(z) \right).$$

To define $g_k : I \rightarrow I$ and $\Psi : \mathbb{N} \times I \rightarrow I$, let $\{r_i\}_{i=1}^\infty = \mathbb{Q}|_{(0,1)}$ be a sequence of all rationals in $(0, 1)$, and $m_1 < m_2 < m_3 < \dots$ an increasing sequence of integers which we specify later. Then

$$g_k = \begin{cases} h_l & \text{if } m_{3l+1} \leq k < m_{3l+2} \\ Id & \text{if } m_{3l+2} \leq k < m_{3l+3} \\ h_l^{-1} & \text{if } m_{3l+3} \leq k < m_{3l+4} \end{cases} \quad k, l \in \mathbb{N}_0 \quad (5)$$

where $h_l : I \rightarrow I$ is a continuous strictly increasing mapping with three fixed points $0, 1, r_l$ and

$$\lim_{l \rightarrow \infty} \|h_l - Id\| = 0; \quad h_l(x) < x \text{ for } x \in (0, r_l); \quad h_l(x) > x \text{ for } x \in (r_l, 1).$$

The sequence $\{m_i\}_{i=1}^\infty$ is defined in the following way:

$$m_{3l+2} - m_{3l+1} = m_{3l+4} - m_{3l+3} = n_l,$$

where n_l is integer satisfying

$$h_l^{n_l}([0, r_l - \frac{1}{l}]) \subset [0, \frac{1}{l}] \quad \wedge \quad h_l^{n_l}([r_l + \frac{1}{l}, 1]) \subset (1 - \frac{1}{l}, 1], \quad (6)$$

and simultaneously $\{m_i\}_{i=1}^\infty$ can be chosen such that

$$m_{3l+3} - m_{3l+2} > \frac{2l}{\epsilon_l}, \quad \text{where } \epsilon_l = \min\{h_l^{n_l}(\frac{1}{l}), 1 - h_l^{n_l}(1 - \frac{1}{l})\}, \quad (7)$$

$$\begin{aligned} \lim_{l \rightarrow \infty} \frac{m_{3l+1}}{m_{3l+2}} &= \lim_{l \rightarrow \infty} \frac{m_{3l+3}}{m_{3l+4}} = 1, \\ \lim_{l \rightarrow \infty} \frac{m_{3l+2}}{m_{3l+3}} &= 0. \end{aligned} \quad (8)$$

The angle of rotation $\Psi : \mathbb{N} \times I \rightarrow I$ is defined as

$$\Psi(k, z) = \begin{cases} z & \text{if } 1 \leq k < m_4 \\ z/l & \text{if } m_{3l+1} \leq k < m_{3l+4} \end{cases} \quad l \in \mathbb{N}.$$

The factor space Y is simply X with fixed $\phi = 0$, i.e. for each point $y \in Y$,

$$y = [2 - \frac{1}{k}, 0, z] \quad \text{or} \quad [2, 0, z], \quad k \in \mathbb{N}, z \in I.$$

To simplify the notation, we skip the second zero coordinate and treat Y as a two-dimensional space. The space Y is union of converging sequence of unit fibers and the limit fiber,

$$Y = \{2 - \frac{1}{k} : k \in \mathbb{N}\} \times I \cup \{2\} \times I.$$

Then the system (X, F) is semiconjugated with skew-product map $f : Y \rightarrow Y$, which is identity on the limit fiber,

$$[2, z] \mapsto [2, z],$$

and which is g_k on inner fibers,

$$[2 - \frac{1}{k}, z] \mapsto [2 - \frac{1}{k+1}, g_k(z)], \quad k \in \mathbb{N}.$$

I. The factor system (Y, f) is DC1.

We show that set $S = \{1\} \times I$ is a distributionally scrambled set, i.e. for any pair of distinct points $(u, v) \in S^2$,

$$\Phi_{(u,v)}^* \equiv 1 \text{ and } \Phi_{(u,v)}(\epsilon) = 0, \text{ where } \epsilon < 1. \quad (9)$$

Since $\{r_i\}_{i=1}^\infty$ is dense in I and by (6), we can find a sequence $\{s_k\}_{k=1}^\infty$ such that $d(f^i(u), f^i(v)) < \frac{1}{s_k}$, for $m_{3s_k+2} \leq i < m_{3s_k+3}$, and therefore, by (8), $\Phi_{(u,v)}^* \equiv 1$. Suppose $u^2 > v^2$, where x^2 denotes the second coordinate of a point x . We can find another subsequence $\{q_k\}_{k=1}^\infty$ such that $d(f^i(u), f^i([1, 1])) < \frac{1}{q_k}$ and simultaneously $d(f^i(v), f^i([1, 0])) < \frac{1}{q_k}$, for $m_{3q_k+2} \leq i < m_{3q_k+3}$. Since f preserves the distance between the endpoints of any fiber, $d(f^i([1, 1]), f^i([1, 0])) = 1$, for $i \geq 0$, we can conclude, by (8), $\Phi_{(u,v)}(\epsilon) = 0$, for any $\epsilon < 1$.

II. (X, F) has no distributionally scrambled pair

We claim $\Phi_{(u,v)}^* < 1$ for any pair of distinct points in X . Let X_0 be the limit cylinder $X_0 = \{[2 \cos 2\pi\phi, 2 \sin 2\pi\phi] : \phi \in I\} \times I$ and $\tilde{X} = X \setminus X_0$. Consider 4 possible cases:

a) $(u, v) \in \tilde{X}$ with $z_u = z_v = z$, $k_u = k_v = k$, $\phi_u \neq \phi_v$.

The angle of rotation is the same for both u and v , $\Psi(k_u, z_u) = \Psi(k_v, z_v) = \Psi(k, z)$, hence, by (4),

$$d(F(u), F(v)) = \rho(\phi_u + \Psi(k, z), \phi_v + \Psi(k, z)) = \rho(\phi_u, \phi_v) = d(u, v).$$

F is isometric in this case and $\Phi_{(u,v)}^* \neq 1$.

b) $(u, v) \in \tilde{X}$ with $z_u \neq z_v$, $k_u = k_v = k$, $\phi_u \neq \phi_v$.

Without loss of generality suppose $k = 1$ (otherwise consider the pre images $(F^{-k}(u), F^{-k}(v))$) and let L be an integer such that $|z_u - z_v| > \frac{1}{L}$. It is sufficient to show that there is $0 < \delta < \frac{1}{3}$, for which

$$\frac{1}{m_{3l+3} - m_{3l+2}} \#\{m_{3l+2} < i < m_{3l+3}; d(F^i(u), F^i(v)) < \delta\} < 3\delta, \quad \text{for any } L \leq l.$$

Since d is max-metric, it is sufficient to prove

$$\frac{1}{m_{3l+3} - m_{3l+2}} \#\{m_{3l+2} < i < m_{3l+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta.$$

Since $|h_L^{n_L}(z_u) - h_L^{n_L}(z_v)| > \epsilon_L$ (see (5) and definition of ϵ_L in (10)), and $|h_L^{n_L}(z_u) - h_L^{n_L}(z_v)|$ is the minimal distance between trajectories of u and v between times m_{3L+1} and m_{3L+4} , it follows

$$\min_{3L+1 < k \leq 3L+4} |g_k \circ g_{k-1} \circ \dots \circ g_{3L+1}(z_u) - g_k \circ g_{k-1} \circ \dots \circ g_{3L+1}(z_v)| > \epsilon_L. \quad (10)$$

Denote the relative angle of rotation of points with height z_u and z_v in the k -th cylinder by $\Delta\Psi_k(z_u, z_v) = |\Psi(k, z_u) - \Psi(k, z_v)| = \frac{|z_u - z_v|}{L}$, for $m_{3L+1} \leq k < m_{3L+4}$. By (10),

$$\Delta\Psi_k(g_k \circ g_{k-1} \circ \dots \circ g_{3L+1}(z_u), g_k \circ g_{k-1} \circ \dots \circ g_{3L+1}(z_v)) > \frac{\epsilon_L}{L}, \quad \text{for } m_{3L+1} \leq k < m_{3L+4}.$$

Since $m_{3L+3} - m_{3L+2} > \frac{2L}{\epsilon_L}$, we can use Lemma 1 and conclude, for any $\delta > 0$,

$$\frac{1}{m_{3L+3} - m_{3L+2}} \#\{m_{3L+2} < i < m_{3L+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta.$$

We obtain the result for any $l > L$ using the same argument, since for every $l > L$, $|z_u - z_v| > \frac{1}{l}$.

c) $(u, v) \in \tilde{X}$ with $z_u \neq z_v$, $k_u \neq k_v$, $\phi_u \neq \phi_v$.

Without loss of generality suppose $k_u = 1$ and $k_v = p$. If $|z_u - z_v| > \frac{1}{L}$, then by case b)

$$\#\{m_{4L+2} + p < i < m_{4L+3} - p; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta \cdot (m_{3L+3} - m_{3L+2})$$

and hence

$$\frac{1}{m_{3L+3} - m_{3L+2}} \#\{m_{3L+2} < i < m_{3L+3}; \rho(\phi_{F^i(u)}, \phi_{F^i(v)}) < \delta\} < 3\delta + \frac{2p}{m_{3L+3} - m_{3L+2}} < 1,$$

for sufficiently large L .

d) $u \in \tilde{X}$ and $v \in X_0$

Since $v \in X_0$ is fixed and $\phi_v = \phi_{F(v)}$, we can find another point in \tilde{X} , $w = [(2 - \frac{1}{k_u}) \cos 2\pi\phi_v, (2 - \frac{1}{k_u}) \sin 2\pi\phi_v, 0]$, which is also fixed under rotation. Therefore

$$\rho(\phi_{F(u)}, \phi_{F(v)}) = \rho(\phi_{F(u)}, \phi_{F(w)})$$

and we can apply case b) or c) to investigate the pair (u, v) instead of (u, w) . \square

Remark. Notice that the upper distributional function for the extension remains positive, $\Phi_{(u,v)}^* > 0$, for any pair of distinct points in $S \times \{1\}$. By (9), $\Phi_{(u,v)} < \Phi_{(u,v)}^*$, hence the system (X, F) is distributionally chaotic of type 3. This fact implies an open question: *Is there a DC3 system which is semiconjugated to an extension without any distributionally scrambled pairs of type 3?*

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